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# Projective Group Representations in Quaternionic Hilbert Space

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## ABSTRACT

We extend the discussion of projective group representations in quaternionic Hilbert space which was given in our recent book. The associativity condition for quaternionic projective representations is formulated in terms of unitary operators and then analyzed in terms of their generator structure. The multi-centrality and centrality assumptions are also analyzed in generator terms, and implications of this analysis are discussed.

# I. ASSOCIATIVITY CONDITION FOR QUATERNIONIC PROJECTIVE GROUP REPRESENTATIONS

In quaternionic quantum mechanics, all symmetries of the transition probabilities are generated by unitary transformations acting on the states of Hilbert space.<sup>1-3</sup> In the simplest case, the unitary transformations  $U_a, U_b, \dots$  form a representation (or vector representation) of the symmetry group with elements  $a, b, \dots$ ,

$$U_b U_a = U_{ba} \quad . \quad (1)$$

A more general possibility is that the group multiplication table is represented over the *rays* corresponding to a complete set of physical states, but not over individual state vectors chosen as ray representatives. This more general composition rule defines a quaternionic *projective representation* (or ray representation), and takes the form (Adler<sup>4</sup>, Sec. 4.3)

$$U_b U_a |f\rangle = U_{ba} |f\rangle \omega(f; b, a), \quad |\omega(f; b, a)| = 1 \quad , \quad (2)$$

for one particular complete set of states  $|f\rangle$  and a set of quaternionic phases  $\omega(f; b, a)$ . When we change ray representative from  $|f\rangle$  to  $|f_\phi\rangle \equiv |f\rangle\phi$ , with  $|\phi| = 1$ , the phase defining the projective representation is easily seen to transform as

$$\omega(f_\phi; b, a) = \bar{\phi} \omega(f; b, a) \phi \quad , \quad (3)$$

with the bar denoting quaternion conjugation. Equation (3) shows clearly that the projective phase  $\omega$  must depend on the state label  $f$  as well as on the group elements  $a, b$ ; failure to take this into account can lead<sup>4</sup> to erroneous conclusions (as in <sup>5</sup>) concerning quaternionic projective representations.

The defining relation for quaternionic projective representations given in Eq. (2) can be rewritten in operator form by defining a left-acting operator  $\Omega(b, a)$ ,

$$\Omega(b, a) = \sum_f |f\rangle \omega(f; b, a) \langle f| \quad , \quad (4a)$$

which using Eq. (3) is seen to be independent of the ray representative chosen for the states  $|f\rangle$ . Multiplying Eq. (2) from the right by  $\langle f|$  and summing over the complete set of states  $|f\rangle$ , we obtain the operator form of the projective representation,

$$U_b U_a = U_{ba} \Omega(b, a) \quad . \quad (4b)$$

It is also immediate from the definition of Eq. (4a), and the fact that  $|\omega| = 1$ , that the operator  $\Omega(b, a)$  is quaternion unitary,

$$\Omega(b, a)^\dagger \Omega(b, a) = \Omega(b, a) \Omega(b, a)^\dagger = 1 \quad . \quad (5)$$

Note that if we were to make the definition of a quaternionic projective representation more restrictive by requiring that Eq. (2) hold for *all* states in Hilbert space, rather than for one particular complete set of states, then we would require  $\Omega(b, a) = 1$ , since the unit operator is the only unitary operator which is simultaneously diagonal on all complete bases in quaternionic Hilbert space. Hence this more restrictive definition excludes quaternionic embeddings of complex projective representations, whereas these are admitted as quaternionic projective representations by the definition of Eq. (2).

A nontrivial condition on the projective representation structure is obtained from the associativity of multiplication in quaternionic Hilbert space, which implies

$$(U_c U_b) U_a = U_c (U_b U_a) \quad . \quad (6)$$

Applying Eq. (4b) twice to the left hand side of Eq. (6), we get

$$\begin{aligned} (U_c U_b) U_a &= U_{cb} \Omega(c, b) U_a = U_{cb} U_a U_a^{-1} \Omega(c, b) U_a \\ &= U_{cba} \Omega(cb, a) U_a^{-1} \Omega(c, b) U_a \quad , \end{aligned} \quad (7a)$$

while applying Eq. (4b) twice to the right hand side of Eq. (6) gives

$$\begin{aligned} U_c (U_b U_a) &= U_c U_{ba} \Omega(b, a) \\ &= U_{cba} \Omega(c, ba) \Omega(b, a) \quad . \end{aligned} \quad (7b)$$

Upon multiplying from the left by  $U_{cba}^{-1}$ , Eqs. (7a,b) give the operator form of the *associativity condition*

$$\Omega(c, ba) \Omega(b, a) = \Omega(cb, a) U_a^{-1} \Omega(c, b) U_a \quad . \quad (8)$$

We can also express the associativity condition as a condition on the quaternionic phase  $\omega(f; b, a)$  introduced in Eq. (2), by applying the spectral representation of Eq. (4a) to the operator form of the associativity condition given in Eq. (8). From Eq. (4a) we get

$$\Omega(c, ba) = \sum_f |f\rangle \omega(f; c, ba) \langle f| \quad , \quad (9a)$$

which when multiplied from the right by Eq. (4a) gives

$$\Omega(c, ba) \Omega(b, a) = \sum_f |f\rangle \omega(f; c, ba) \omega(f; b, a) \langle f| \quad . \quad (9b)$$

Equation (4a) and the unitarity of  $\Omega(cb, a)$  also imply that

$$\Omega(cb, a)^{-1} = \sum_f |f\rangle \overline{\omega(f; cb, a)} \langle f| \quad , \quad (9c)$$

and so the associativity condition of Eq. (8) can be rewritten as

$$\begin{aligned} U_a^{-1} \Omega(c, b) U_a &= \Omega(c, ba) \Omega(b, a) \Omega(cb, a)^{-1} \\ &= \sum_f |f\rangle \omega(f; c, ba) \omega(f; b, a) \overline{\omega(f; cb, a)} \langle f| \quad . \end{aligned} \quad (10)$$

Hence  $U_a^{-1}\Omega(c, b)U_a$  is diagonal in the basis spanned by the states  $|f\rangle$ . Taking matrix elements of Eq. (10), and using the unitarity of  $U_a$ , the associativity condition gives the two relations

$$\omega(f; c, ba)\omega(f; b, a)\overline{\omega(f; cb, a)} = \sum_{f''} \overline{\langle f''|U_a|f\rangle}\omega(f''; c, b)\langle f''|U_a|f\rangle \quad , \quad (11)$$

and, when  $\langle f'|f\rangle = 0$ ,

$$0 = \sum_{f''} \overline{\langle f''|U_a|f\rangle}\omega(f''; c, b)\langle f''|U_a|f'\rangle \quad . \quad (12)$$

We conclude this section by comparing the quaternionic Hilbert space form of the associativity condition with the simpler form which is familiar from complex Hilbert space.<sup>6,7</sup> In a complex Hilbert space, the phase  $\omega(f; b, a)$  introduced in Eq. (2) is a complex number, and commutes with the phase  $\phi$ , also now complex, which we introduced in Eq. (3) to describe a change of ray representative. Hence Eq. (3) implies, in the complex case, that  $\omega(f; b, a)$  is independent of the ray representative chosen for the state  $|f\rangle$ , and it is then consistent to assume that  $\omega(f; b, a)$  is independent of the state label  $f$ , so that

$$\omega(f; b, a) = \omega(b, a) \quad \text{complex Hilbert space} \quad . \quad (13a)$$

Substituting Eq. (13a) into Eq. (4a), we now get

$$\Omega(b, a) = \sum_f |f\rangle\omega(b, a)\langle f| = \omega(b, a) \sum_f |f\rangle\langle f| = \omega(b, a)1 \quad , \quad (13b)$$

where 1 denotes the unit operator in complex Hilbert space. Since the complex phase  $\omega(b, a)$  is a  $c$ -number in complex Hilbert space, on substituting Eq. (13b) into Eq. (4b) we learn that

$$U_b U_a = U_{ba}\omega(b, a) = \omega(b, a)U_{ba} \quad , \quad (14a)$$

which is the standard definition of a projective representation in complex Hilbert space. Moreover, since Eq. (13b) implies that  $\Omega(b, a)$  commutes with the unitary operator  $U_a$ , the associativity condition of Eqs. (8) and (11) reduces to the familiar complex Hilbert space form

$$\omega(c, ba)\omega(b, a) = \omega(cb, a)\omega(c, b) \quad . \quad (14b)$$

## II. THE ASSOCIATIVITY CONDITION IN GENERATOR FORM

Let us now assume that the symmetry group with which we are dealing is a Lie group, so that in the neighborhood of the identity  $e$  the unitary transformations  $U_a$ ,  $U_b$ ,  $U_{ba}$ , ... can be written in terms of a set of anti-self-adjoint generators  $\tilde{G}_A$  as

$$U_a = \exp\left(\sum_A \theta_A^a \tilde{G}_A\right), \quad U_b = \exp\left(\sum_A \theta_A^b \tilde{G}_A\right), \quad U_{ba} = \exp\left(\sum_A \theta_A^{ba} \tilde{G}_A\right), \dots, \quad (15a)$$

with  $\theta_A^e = 0$  and  $U_e = 1$ . Then Eq. (4b) implies that  $\Omega(b, a)$  must be unity when either  $a$  or  $b$  is the identity, and thus the generator form for this operator is

$$\Omega(b, a) = \exp\left(\frac{1}{2} \sum_{BA} [\theta_B^b \theta_A^a \tilde{I}_{BA} + \sum_C \theta_B^b \theta_C^b \theta_A^a \tilde{J}_{(BC)A}^{(1)} + \sum_C \theta_B^b \theta_A^a \theta_C^a \tilde{J}_{B(AC)}^{(2)} + O(\theta^4)]\right), \quad (15b)$$

where the parentheses ( ) around a set of indices indicate that the tensor in question is symmetric in those indices, and where we use the tilde to indicate operators which are anti-self-adjoint. The parameters  $\theta_C^{ba}$  must be functions of the parameters  $\theta_A^a$  and  $\theta_B^b$ ,

$$\theta_C^{ba} = \psi_C^{ba}(\{\theta_B^b\}, \{\theta_A^a\}) = \theta_C^b + \theta_C^a + \frac{1}{2} \sum_{BA} C_{BAC} \theta_B^b \theta_A^a + O(\theta^3), \quad (15c)$$

where in making the Taylor expansion we have used the fact that  $U_{be} = U_b$  and  $U_{ea} = U_a$ , which fixes the linear terms in the expansion and requires the quadratic term to be bilinear.

We proceed now to derive a number of relations by combining the generator expansions of Eqs. (15a–c) with the formulas of Sec. I. We begin by substituting Eqs. (15a–c) into Eq. (4b) using the Baker–Campbell– Hausdorff formula,

$$\exp X \exp Y = \exp(X + Y + \frac{1}{2}[X, Y] + \dots) \quad , \quad (16a)$$

to combine exponents arising from the factors on the left and right. From the left hand side of Eq. (4b) we get

$$U_b U_a = \exp\left(\sum_B \theta_B^b \tilde{G}_B + \sum_A \theta_A^a \tilde{G}_A + \frac{1}{2} \sum_{BA} \theta_B^b \theta_A^a [\tilde{G}_B, \tilde{G}_A] + O(\theta^3)\right) \quad , \quad (16b)$$

while from the right hand side of Eq. (4b) we get

$$U_{ba} \Omega(b, a) = \exp\left(\sum_C (\theta_C^b + \theta_C^a) \tilde{G}_C + \frac{1}{2} \sum_{CBA} C_{BAC} \theta_B^b \theta_A^a \tilde{G}_C + \frac{1}{2} \sum_{BA} \theta_B^b \theta_A^a \tilde{I}_{BA} + O(\theta^3)\right) \quad . \quad (16c)$$

Equating Eqs. (16b) and (16c) thus gives the relations

$$[\tilde{G}_B, \tilde{G}_A] = \sum_C C_{[BA]C} \tilde{G}_C + \tilde{I}_{[BA]} \quad (17a)$$

and

$$0 = \sum_C C_{(BA)C} \tilde{G}_C + \tilde{I}_{(BA)} \quad , \quad (17b)$$

where the square brackets [ ] around a set of indices indicates that the tensor in question is antisymmetric in these indices. We shall restrict ourselves henceforth to the case in which  $C_{(BA)C} = 0$ , which by Eq. (17b) implies that  $\tilde{I}_{(BA)} = 0$ ; making this assumption then implies that  $C_{BAC} = C_{[BA]C}$  and  $\tilde{I}_{BA} = \tilde{I}_{[BA]}$ . In other words, we are assuming that the structure constants  $C_{BAC}$  for a projective representation have the same antisymmetric form as holds for a vector representation. Changing the summation index  $C$  to  $D$  in Eq. (17a), and then



taking the commutator of Eq. (17a) with  $\tilde{G}_C$ , we find

$$[\tilde{G}_C, [\tilde{G}_B, \tilde{G}_A]] = \sum_D C_{[BA]D} [\tilde{G}_C, \tilde{G}_D] + [\tilde{G}_C, \tilde{I}_{[BA]}] \quad ; \quad (18a)$$

adding to this identity the two related identities obtained by cyclically permuting  $A, B, C$ , using the fact that the left hand side of the sum vanishes by the Jacobi identity for the commutator, and substituting Eq. (17a) for the commutators appearing on the right hand side of the sum, we get the identity

$$\begin{aligned} & \sum_{DE} (C_{[BA]D} C_{[CD]E} + C_{[CB]D} C_{[AD]E} + C_{[AC]D} C_{[BD]E}) \tilde{G}_E \\ & + \sum_D (C_{[BA]D} \tilde{I}_{[CD]} + C_{[CB]D} \tilde{I}_{[AD]} + C_{[AC]D} \tilde{I}_{[BD]}) \\ & + [\tilde{G}_C, \tilde{I}_{[BA]}] + [\tilde{G}_A, \tilde{I}_{[CB]}] + [\tilde{G}_B, \tilde{I}_{[AC]}] = 0 \quad . \end{aligned} \quad (18b)$$

We next substitute Eqs. (15a–c) into the associativity condition of Eq. (8), now keeping cubic terms in the exponent of the form  $\theta_A^a \theta_B^b \theta_C^c$ , but dropping cubic terms, such as  $\theta_A^a \theta_B^a \theta_C^c$ , that do not contain all three of the upper indices  $a, b, c$ . For the first factor on the left hand side of Eq. (8), we find from Eqs. (15b) and (15c) that

$$\begin{aligned} \Omega(c, ba) &= \exp \left( \frac{1}{2} \sum_{BA} (\theta_B^c \theta_A^{ba} \tilde{I}_{[BA]} + \sum_C \theta_B^c \theta_A^{ba} \theta_C^{ba} \tilde{J}_{B(AC)}^{(2)}) \right) \\ &= \exp \left( \frac{1}{2} \sum_{BA} [\theta_B^c (\theta_A^b + \theta_A^a + \frac{1}{2} \sum_{DE} C_{[DE]A} \theta_D^b \theta_E^a) \tilde{I}_{[BA]} + 2 \sum_C \theta_B^c \theta_A^b \theta_C^a \tilde{J}_{B(AC)}^{(2)}] \right) \quad , \end{aligned} \quad (19a)$$

while for the second factor on the left hand side of Eq. (8) we have

$$\Omega(b, a) = \exp \left( \frac{1}{2} \sum_{BA} \theta_B^b \theta_A^a \tilde{I}_{[BA]} \right) \quad . \quad (19b)$$

Since the exponents in Eqs. (19a, b) both begin at order  $\theta^2$ , through order  $\theta^3$  we can simply add exponents to get the product on the left hand side of Eq. (8). Proceeding similarly for

the first factor on the right hand side of Eq. (8), we get

$$\begin{aligned}\Omega(cb, a) &= \exp \left( \frac{1}{2} \sum_{BA} (\theta_B^{cb} \theta_A^a \tilde{I}_{[BA]} + \sum_C \theta_B^{cb} \theta_C^{cb} \theta_A^a \tilde{J}_{(BC)A}^{(1)}) \right) \\ &= \exp \left( \frac{1}{2} \sum_{BA} [(\theta_B^c + \theta_B^b + \frac{1}{2} \sum_{DE} C_{[DE]B} \theta_D^c \theta_E^b) \theta_A^a \tilde{I}_{[BA]} + 2 \sum_C \theta_B^c \theta_C^b \theta_A^a \tilde{J}_{(BC)A}^{(1)}] \right) \quad ,\end{aligned}\tag{20a}$$

while for the second factor on the right hand side of Eq. (8), use of the Baker–Campbell–Hausdorff formula gives

$$\begin{aligned}U_a^{-1} \Omega(c, b) U_a &= \exp(-\sum_A \theta_A^a \tilde{G}_A) \exp(\frac{1}{2} \sum_{CB} \theta_C^c \theta_B^b \tilde{I}_{[CB]}) \exp(\sum_A \theta_A^a \tilde{G}_A) \\ &= \exp(\frac{1}{2} \sum_{CB} \theta_C^c \theta_B^b \tilde{I}_{[CB]} - \frac{1}{2} \sum_A \sum_{CB} \theta_A^a \theta_C^c \theta_B^b [\tilde{G}_A, \tilde{I}_{[CB]}]) \quad .\end{aligned}\tag{20b}$$

Since the exponents in Eqs. (20a, b) begin at order  $\theta^2$ , it again suffices to simply add the exponents to form the product appearing on the right hand side of Eq. (8). Thus, to the requisite order, the content of Eq. (8) is obtained by equating the sum of the exponents in Eqs. (19a, b) to the corresponding sum of exponents in Eqs. (20a, b). The quadratic terms in  $\theta$  are immediately seen to be identical on left and right, while the cubic term proportional to  $\theta_A^a \theta_B^b \theta_C^c$  gives (after some relabeling of dummy summation indices) the nontrivial identity

$$\tilde{J}_{C(BA)}^{(2)} + \frac{1}{4} \sum_D C_{[BA]D} \tilde{I}_{[CD]} = \tilde{J}_{(CB)A}^{(1)} + \frac{1}{4} \sum_D C_{[CB]D} \tilde{I}_{[DA]} - \frac{1}{2} [\tilde{G}_A, \tilde{I}_{[CB]}] \quad .\tag{21}$$

On totally antisymmetrizing with respect to the indices  $A, B, C$ , the terms in Eq. (21) involving  $\tilde{J}^{(1,2)}$  drop out, and we are left with the identity

$$\begin{aligned}&\sum_D (C_{[BA]D} \tilde{I}_{[CD]} + C_{[CB]D} \tilde{I}_{[AD]} + C_{[AC]D} \tilde{I}_{[BD]}) \\ &+ [\tilde{G}_C, \tilde{I}_{[BA]}] + [\tilde{G}_A, \tilde{I}_{[CB]}] + [\tilde{G}_B, \tilde{I}_{[AC]}] = 0 \quad .\end{aligned}\tag{22a}$$

In other words, associativity implies that the sum of the second and third lines of Eq. (18b) vanishes separately; hence the first line of Eq. (18b) must also vanish, and since the generators

$\tilde{G}_E$  are linearly independent this gives the Jacobi identity for the structure constants,

$$\sum_{DE} (C_{[BA]D}C_{[CD]E} + C_{[CB]D}C_{[AD]E} + C_{[AC]D}C_{[BD]E}) = 0 \quad . \quad (22b)$$

In the complex case, in which  $\Omega(a, b) = \omega(a, b)1$  is a  $c$ -number, the tensor  $\tilde{I}_{[AB]}$  is a  $c$ -number “central charge” and the commutator terms in Eqs. (18b) and (22a) vanish identically. Therefore, in the complex case, Eq. (18b) implies both Eq. (22b) and the identity

$$\sum_D (C_{[BA]D}\tilde{I}_{[CD]} + C_{[CB]D}\tilde{I}_{[AD]} + C_{[AC]D}\tilde{I}_{[BD]}) = 0 \quad \text{complex case}, \quad (23)$$

and so one obtains the entire content of the associativity condition from the simpler analysis leading to Eq. (18b), without having to perform the third order expansion needed to get Eq. (22a).

### III. GENERAL, MULTI-CENTRAL, AND CENTRAL QUATERNIONIC PROJECTIVE REPRESENTATIONS

The analysis of Sec. II applies to the general case (apart from the restriction  $C_{(BA)C} = 0$ ) of a quaternionic projective representation; in order to obtain more detailed results it is necessary to introduce further structural assumptions. In Ref. 4 two special classes of quaternionic projective representations are defined. A quaternionic projective representation is defined to be *multi-central* if

$$[\Omega(b, a), U_a] = [\Omega(b, a), U_b] = 0 \quad , \quad \text{all } a, b \quad , \quad (24a)$$

while it is defined to be *central* if

$$[\Omega(b, a), U_c] = 0 \quad , \quad \text{all } a, b, c \quad . \quad (24b)$$

Expressed in terms of the generators introduced in Eqs. (15a, b), the multi-centrality con-

dition takes the form

$$\sum_{ABC} \theta_A^a \theta_B^b \theta_C^a [\tilde{G}_C, \tilde{I}_{[BA]}] = \sum_{ABC} \theta_A^a \theta_B^b \theta_C^b [\tilde{G}_C, \tilde{I}_{[BA]}] = 0, \quad \text{all } a, b, \quad (25a)$$

while the centrality condition becomes

$$\sum_{ABC} \theta_A^a \theta_B^b \theta_C^c [\tilde{G}_C, \tilde{I}_{[BA]}] = 0, \quad \text{all } a, b, c. \quad (25b)$$

Making the definition

$$\Delta_{[AB]C} = [\tilde{G}_C, \tilde{I}_{[BA]}], \quad (25c)$$

we see from Eq. (25a) that multi-centrality requires that  $\Delta_{[AB]C}$  be antisymmetric in  $A, C$  and in  $B, C$  as well as in  $A, B$ ; thus in the multi-central case  $\Delta$  is totally antisymmetric, which we will indicate by writing it as  $\Delta_{[ABC]}$ . From Eq. (25b), we see that centrality requires that  $\Delta_{[AB]C}$  must vanish.

Using the generator formulation, we proceed now to discuss successively the general, multi-central, and central cases in the light of the associativity analysis of Sec. II.

(1) *The general case.* An example given in Eqs. (13.54g) and (14.23a) of Ref. 4 shows that one can have a quaternionic projective representation which is neither central nor multi-central. The example is constructed from  $n$  independent fermion creation and annihilation operators  $b_\ell^\dagger, b_\ell, \ell = 1, \dots, n$ , which commute with a left algebra quaternion basis  $E_0 = 1, E_1 = I, E_2 = J, E_3 = K$ . Consider the set of three generators  $\tilde{G}_A$  defined by

$$\tilde{G}_A = -\frac{1}{2} E_A N, \quad A = 1, 2, 3, \quad (26a)$$

with  $N$  the number operator

$$N = \sum_{\ell=1}^n b_\ell^\dagger b_\ell. \quad (26b)$$

The commutator algebra of these generators has the form of a projective representation of  $SU(2)$ ,

$$\begin{aligned} [\tilde{G}_B, \tilde{G}_A] &= - \sum_{C=1}^3 \epsilon_{[BAC]} \tilde{G}_C + \tilde{I}_{[BA]} , \\ \tilde{I}_{[BA]} &= \sum_{C=1}^3 \epsilon_{[BAC]} \frac{1}{2} E_C N(N-1) , \end{aligned} \quad (26c)$$

with  $\epsilon$  the usual three index antisymmetric tensor. A simple calculation now shows that

$$[\tilde{G}_A, \tilde{I}_{[BC]}] = -N(N-1)(\delta_{AB}\tilde{G}_C - \delta_{AC}\tilde{G}_B) , \quad (27a)$$

which is not antisymmetric in either the index pair  $A, C$  or the pair  $A, B$ , and so the multi-centrality condition is not satisfied. Another simple calculation shows that

$$\sum_D (\epsilon_{[BAD]} \tilde{I}_{[CD]} + \epsilon_{[CBD]} \tilde{I}_{[AD]} + \epsilon_{[ACD]} \tilde{I}_{[BD]}) = 0 , \quad (27b)$$

by virtue of the Jacobi identity for the structure constant  $\epsilon$ , and also

$$[\tilde{G}_C, \tilde{I}_{[BA]}] + [\tilde{G}_A, \tilde{I}_{[CB]}] + [\tilde{G}_B, \tilde{I}_{[AC]}] = 0 . \quad (27c)$$

Hence the associativity condition of Eq. (22a) is satisfied, with the first and second lines each vanishing separately.

(2) *The multi-central case.* Let us now consider the multi-central case, in which  $\Delta_{[AB]C}$  defined in Eq. (25c) is totally antisymmetric in  $A, B, C$ , as indicated by the notation  $\Delta_{[ABC]}$ .

The associativity condition of Eq. (22a) then simplifies to

$$\sum_D (C_{[BA]D} \tilde{I}_{[CD]} + C_{[CB]D} \tilde{I}_{[AD]} + C_{[AC]D} \tilde{I}_{[BD]}) + 3\Delta_{[ABC]} = 0 . \quad (28a)$$

A further equation involving  $\Delta$  is obtained from the Jacobi identity

$$[\tilde{G}_D, [\tilde{G}_C, \tilde{I}_{[BA]}]] - [\tilde{G}_C, [\tilde{G}_D, \tilde{I}_{[BA]}]] = [\tilde{I}_{[BA]}, [\tilde{G}_C, \tilde{G}_D]] , \quad (28b)$$

which on substituting Eqs. (17a) and (25c) becomes

$$[\tilde{G}_D, \Delta_{[AB]C}] - [\tilde{G}_C, \Delta_{[AB]D}] = - \sum_E C_{[CD]E} \Delta_{[AB]E} + [\tilde{I}_{[BA]}, \tilde{I}_{[CD]}] \quad , \quad (28c)$$

an equation which holds even in the general case in which  $\Delta$  is not totally antisymmetric. Specializing Eq. (28c) to the multi-central case and contracting it with  $\delta_{AC}\delta_{BD}$ , the left hand side vanishes because of the antisymmetry of  $\Delta$ , while the commutator term on the right hand side becomes  $\sum_{AB} [\tilde{I}_{[BA]}, \tilde{I}_{[AB]}] = 0$ , leaving the identity (after relabeling the dummy index  $E$  as  $C$ )

$$\sum_{ABC} C_{[AB]C} \Delta_{[ABC]} = 0 \quad . \quad (29)$$

Thus in order for a multi-central projective representation to exist which has  $\Delta \neq 0$  and so is not also central, there must be a three index antisymmetric tensor  $\Delta_{[ABC]}$  which vanishes when all three indices are contracted with the structure constant  $C_{[AB]C}$ . This condition is not easy to satisfy and so we pose the question, which we have not been able to answer: Can one construct an example of a multi-central quaternionic projective representation which is not central, or can one prove (in general, or with a restriction, e.g., to simple or semi-simple groups) that a multi-central quaternionic projective representation must always be central? The application of multi-centrality in Ref. 4 sheds no light on this issue; multi-centrality was used there (e.g. in Sec. 12.3) to show that quaternionic Poincaré group projective representations outside the zero energy sector can always be transformed to complex Poincaré group projective representations, which in the sector continuously connected to the identity are known<sup>8</sup> to be transformable to vector representations.

(3) *The central case.* Let us finally consider the central case in which  $\Delta = 0$ , which by Eqs. (25c) and (28c) implies that  $\tilde{I}_{[BA]}$  commutes with both  $\tilde{G}_C$  and  $\tilde{I}_{[CD]}$  for arbitrary values

of the indices. Thus  $\tilde{I}_{[BA]}$  behaves as a central charge, justifying the name “central” for this case. The various results obtained in Bargmann<sup>6</sup> can be immediately generalized to the quaternionic central case; for example, the analysis of Ref. 6 can be easily extended to show that the central charges associated with a quaternionic central projective representation of a semi-simple Lie group can always be removed by redefinition of the generators; and again, the nontrivial illustration<sup>6</sup> of a complex projective representation, constructed in terms of the phase space translation generators in nonrelativistic quantum mechanics, can be embedded<sup>4</sup> in quaternionic quantum mechanics as a central projective representation.

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